

ORIENTED INVOLUTIONS, SYMMETRIC AND SKEW-SYMMETRIC ELEMENTS IN GROUP RINGS

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ABSTRACT. Let G be a group with involution $*$ and $\sigma: G \rightarrow \{\pm 1\}$ a group homomorphism. The map \sharp that sends $\alpha = \sum \alpha_g g$ in a group ring RG to $\alpha^\sharp = \sum \sigma(g) \alpha_g g^*$ is an involution of RG called an *oriented group involution*. An element $\alpha \in RG$ is *symmetric* if $\alpha^\sharp = \alpha$ and *skew-symmetric* if $\alpha^\sharp = -\alpha$. The sets of symmetric and skew-symmetric elements have received a lot of attention in the special cases that $*$ is the inverse map on G and/or σ is identically 1, but not in general. In this paper, we determine the conditions under which the sets of elements that are symmetric and skew-symmetric, respectively, relative to a general oriented involution form subrings of RG . The work on symmetric elements is a modification and correction of previous work.

1. INTRODUCTION

Let $g \mapsto g^*$ denote an involution on a group G , let $\sigma: G \rightarrow \{\pm 1\}$ be a group homomorphism and let R be a commutative ring with 1. For an element $\alpha = \sum_{g \in G} \alpha_g g$ in the group ring RG , define

$$\alpha^\sharp = \sum_{g \in G} \sigma(g) \alpha_g g^*.$$

The map $\alpha \mapsto \alpha^\sharp$ is an involution of RG called an *oriented group involution*, the function σ being the *orientation*. In the case that the involution on G is the classical involution, $g \mapsto g^{-1}$, the map \sharp is precisely the oriented involution introduced by S. P. Novikov in the context of K -theory [Nov70].

Oriented group involutions have been studied for a long time, perhaps starting with the fundamental paper of Giambruno and Sehgal [GS93], but primarily in the special cases that $g^* = g^{-1}$ or $\sigma(g) = 1$ for all $g \in G$. Much of the work in this area is described in the comprehensive treatise by Gregory Lee [Lee10].

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Throughout this paper, we let

$$(RG)^+ = \{\alpha \in RG \mid \alpha^\sharp = \alpha\} \quad \text{symmetric elements}$$

and

$$(RG)^- = \{\alpha \in RG \mid \alpha^\sharp = -\alpha\} \quad \text{skew-symmetric elements.}$$

As indicated, the elements of $(RG)^+$ are called *symmetric* and the elements of $(RG)^-$ are *skew-symmetric*. Historically, the questions of interest have revolved around the commutativity, anticommutativity and Lie properties of the symmetric and skew-symmetric elements. Some properties of *unitary units* ($\alpha^\sharp = \pm\alpha^{-1}$) have also been considered. One might ask when the sets $(RG)^+$ or $(RG)^-$ are subrings of RG . It is not hard to show that these questions are equivalent, respectively, to asking when the elements that are symmetric relative to \sharp commute and when the elements that are skew-symmetric relative to \sharp anticommute. Observing that $(RG)^+$ is a Jordan algebra under the operation $\alpha \circ \beta = \alpha\beta + \beta\alpha$ and that $(RG)^-$ is a Lie algebra under the Lie bracket $[\alpha, \beta] = \alpha\beta - \beta\alpha$, it is natural to ask when these operations are trivial. These questions are equivalent, respectively, to asking when the elements of $(RG)^+$ anticommute and when the elements of $(RG)^-$ commute. All these questions have been studied in special cases; the first question (when is $(RG)^+$ a subring) was studied in general in [CM06], although this paper contained a slight error which we correct here. We also answer completely the question as to when $(RG)^-$ is a subring, thereby settling the first two of the four questions posed above.¹

In this work, we make no assumptions about $*$ and assume that the orientation σ is not identically 1. This is equivalent to the assumption that $\ker \sigma$ has index 2 in the group G and it implies that the characteristic of the ring is not 2. We assume throughout that σ and $*$ are *compatible* in the sense that $\sigma(g^*) = \sigma(g)$ for all $g \in G$ and remark that this is the same as requiring that both $\ker \sigma$ and its complement in G be invariant under $*$.

It is not hard to find examples of oriented involutions where the orientation and involution are not compatible. For example, in

$$D_n = \langle a, b \mid a^n = b^2 = 1, ba = a^{-1}b \rangle,$$

the dihedral group of order $2n$, the involution defined by $a^* = a^{-1}$, $b^* = ab$ and the orientation σ defined by $\sigma(a) = -1$, $\sigma(b) = 1$ are not compatible (in characteristic different from 2) because $\sigma(ab) = -1$, whereas $\sigma((ab)^*) = \sigma(b^*a^*) = \sigma(aba^{-1}) = \sigma(a^2b) = 1$.

On the other hand, compatibility does not appear to be a strong assumption. It is satisfied with any orientation when $*$ is the classical involution $g \mapsto g^{-1}$ and also when $*$ is transpose and σ is determinant on a group of matrices of determinant ± 1 . It is satisfied with $A^* = A^T$ and $\sigma(A) = \frac{\det A}{|\det A|}$

¹The last two questions are the subject of partially completed research and will be the focus of future papers.

on any group of matrices. Furthermore, computer explorations with various groups suggest an abundance of oriented involutions where the orientation is compatible with the involution, sixteen alone in the dihedral group D_4 of order 8 and twenty-four in the quaternion group of order 8.

A group G is said to be *SLC* if it has a unique nonidentity commutator, which we shall always denote s , and the *limited commutativity* property, which says that the only way two elements $g, h \in G$ can commute is with one of the three elements g, h, gh central. On such a group, the map $*$ defined by

$$(1.1) \quad g^* = \begin{cases} g & \text{if } g \text{ is central} \\ sg & \text{otherwise} \end{cases}$$

is an involution (we refer to this as the *canonical* involution on an SLC group) with which any orientation is surely compatible. We will often use (and sometimes implicitly) the easily checked facts that such s must be central, $s^* = s$ and $s^2 = 1$. We refer the reader to [GJM96], and especially to §III.3.

2. SKEW-SYMMETRIC ELEMENTS ANTICOMMUTE

Let $(RG)^- = \{\alpha \in RG \mid \alpha^\# = -\alpha\}$ denote the set of elements of RG that are skew-symmetric relative to an oriented group involution $\#$. In this section, we determine when this set forms a subring, equivalently, when the elements of $(RG)^-$ anticommute.

This question has been answered in the special case that the orientation is trivial with this result [GM]. In characteristic different from 2, either

- G is abelian and $*$ is the identity, or
- $\text{char } R = 4$, G is abelian and there exists $s \in G$ with $s^2 = 1$ and $g^* = g$ or sg for all $g \in G$, or
- $\text{char } R = 4$, G is nonabelian with a unique nonidentity commutator, s (necessarily central of order 2), and $g^* = g$ or sg for all $g \in G$.

Our theorem is this.

Theorem 2.1. *Let $g \mapsto g^*$ denote an involution on a group G and let $\sigma: G \mapsto \{\pm 1\}$ be an orientation homomorphism which is not identically 1. Let R be a commutative ring with 1 and of characteristic different from 2. For $\alpha = \sum_{g \in G} \alpha_g g$ in the group ring RG , define $\alpha^\# = \sum_{g \in G} \alpha_g \sigma(g) g^*$. Assume $*$ and σ are compatible in the sense that $\sigma(g) = \sigma(g^*)$ for all $g \in G$. If the set $(RG)^-$ of elements of RG which are skew symmetric relative to $\#$ anticommute, then*

- (1) $\text{char } R = 4$, G is abelian, $*$ is the identity on $N = \ker \sigma$ and $x^* \neq x$ for all $x \notin N$,
- or
- (2) $\text{char } R = 4$, G is an SLC group with $*$ the canonical involution and $x^* \neq x$ for all $x \notin N$.

Conversely, if G is a group with an index 2 subgroup N and $\sigma: G \rightarrow \{\pm 1\}$ is the homomorphism with kernel N , then $(RG)^-$ is an anticommutative set given either of these situations.

Proof. We begin by determining the nature of the elements of $(RG)^-$. For this, it is convenient to let $N = \ker \sigma$ and to partition the elements of G into four sets,

$$\begin{aligned} S_1 &= \{n \in N \mid n^* = n\}, \\ S_2 &= \{n \in N \mid n^* \neq n\}, \\ S_3 &= \{x \notin N \mid x^* = x\}, \\ S_4 &= \{x \notin N \mid x^* \neq x\}. \end{aligned}$$

Any $\alpha \in RG$ can be written in the form

$$(2.1) \quad \alpha = \sum_{g \in S_1} \alpha_g g + \sum_{g \in S_2} \alpha_g g + \sum_{g \in S_3} \alpha_g g + \sum_{g \in S_4} \alpha_g g$$

and then

$$\alpha^\sharp = \sum_{g \in S_1} \alpha_g g + \sum_{g \in S_2} \alpha_g g^* - \sum_{g \in S_3} \alpha_g g - \sum_{g \in S_4} \alpha_g g^*.$$

Thus

$$\alpha^\sharp = -\alpha \text{ if and only if } \begin{cases} \alpha_g = -\alpha_g & \text{if } g \in S_1 \\ \alpha_g = -\alpha_{g^*} & \text{if } g \in S_2 \\ \alpha_g = \alpha_g & \text{if } g \in S_4. \end{cases}$$

Thus $(RG)^-$ is spanned over R by elements of the sets

$$\begin{aligned} \mathcal{S}_1 &= \{\alpha_n n \mid n^* = n \in N, 2\alpha_n = 0\}, \\ \mathcal{S}_2 &= \{n - n^* \mid n \in N, n^* \neq n\}, \\ \mathcal{S}_3 &= \{x \notin N \mid x^* = x\}, \\ \mathcal{S}_4 &= \{x + x^* \mid x \notin N, x^* \neq x\} \end{aligned}$$

and, of course, $(RG)^-$ is anticommutative if and only if any two elements from the union of the \mathcal{S}_i anticommute. If this is the case, and if $x \in \mathcal{S}_3$, then $xx = -xx$ implies $2x^2 = 0$, which cannot happen in characteristic different from 2. Thus $\mathcal{S}_3 = \emptyset$ and $x^* \neq x$ for every $x \notin N$. In particular, $*$ is not the identity.

We begin with the converse, showing that in each of the situations described in the theorem, the elements of $(RG)^-$ anticommute.

(1) Suppose G is abelian, $*$ is an involution on G which moves every element outside N but whose restriction to N is the identity and $\text{char } R = 4$. Then $\mathcal{S}_2 = \{0\}$ and $\mathcal{S}_3 = \emptyset$. Elements $\alpha_n n$ and $\alpha_m m$ of \mathcal{S}_1 anticommute because $(\alpha_m m)(\alpha_n n) = \alpha_m \alpha_n mn = -\alpha_m \alpha_n mn$ (using $2\alpha_m = 0$) $= -\alpha_n \alpha_m nm = -(\alpha_n n)(\alpha_m m)$. For a similar reason, any element of \mathcal{S}_1

anticommutes with any element of \mathcal{S}_4 . To see that two elements of \mathcal{S}_4 anticommute, we compute

$$(2.2) \quad (x + x^*)(y + y^*) + (y + y^*)(x + x^*) = 2xy + 2xy^* + 2x^*y + 2x^*y^*$$

with $x \notin N$, $y \notin N$, $x^* \neq x$ and $y^* \neq y$. Since $xy \in N$, we have $xy = (xy)^* = y^*x^* = x^*y^*$ and, similarly, $xy^* = x^*y$, so the right side of (2.2) is $4xy + 4x^*y = 0$.

(2) Suppose $\text{char } R = 4$ and G is an SLC group with canonical involution $*$ that moves every element outside N . Thus $x^* = sx$ if $x \notin N$. Again $\mathcal{S}_3 = \emptyset$, so it suffices to show that any two elements of $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_4$ anticommute. First we note that any element $\alpha_n n$ of \mathcal{S}_1 anticommutes with any element β in the group ring because $n^* = n$ means n is central, so $(\alpha_n n)\beta = \beta(\alpha_n n) = -\beta(\alpha_n n)$ because $2\alpha_n = 0$. Take $n - n^*$ and $m - m^*$ in \mathcal{S}_2 . Then $n - n^* = n - sn = (1-s)n$ and $m - m^* = (1-s)m$. Since $(1-s)^2 = 1 - 2s + s^2 = 2(1-s)$,

$$\begin{aligned} (n - n^*)(m - m^*) &= 2(1-s)nm \\ &= \begin{cases} 2(1-s)mn & \text{if } nm = mn \\ 2(1-s)smn = -2(1-s)mn & \text{if } nm \neq mn \end{cases} \end{aligned}$$

and, because the characteristic is 4, in either case we get $-2(1-s)mn = -(m - m^*)(n - n^*)$. The situation is similar if we consider two elements $x + x^* = (1+s)x$ and $y + y^* = (1+s)y$ of \mathcal{S}_4 because

$$\begin{aligned} (x + x^*)(y + y^*) &= 2(1+s)xy \\ &= \begin{cases} 2(1+s)yx & \text{if } xy = yx \\ 2(1+s)syx = 2(1+s)yx & \text{if } xy \neq yx \end{cases} \end{aligned}$$

and again, because $\text{char } R = 4$, $2(1+s)yx = -2(1+s)yx = -(y + y^*)(x + x^*)$. Finally, we observe that any element of \mathcal{S}_2 anticommutes with any element of \mathcal{S}_4 because the product of two such elements is 0: $(n - n^*)(x + x^*) = (1-s)(1+s)nx = 0$.

Now we attack the first statement of the theorem assuming that the elements of $(RG)^-$ anticommute and hence $\mathcal{S}_3 = \emptyset$.

The compatibility condition implies that N is invariant under $*$, so we may apply to N the results of [GM] mentioned at the start of this section in the case that the orientation is identically 1. There are three possibilities.

1) Suppose first that N is abelian and $*$ is the identity on N . Let $x \notin N$. Since $x^* \neq x$, $x + x^* \in \mathcal{S}_4$. The equation $(x + x^*)(x + x^*) = -(x + x^*)(x + x^*)$ implies $2(x + x^*)^2 = 0$ which, using $(x^*)^2 = (x^2)^* = x^2$ because $x^2 \in N$, gives

$$2(2x^2 + xx^* + x^*x) = 0.$$

Now $x^2 \neq xx^*$ and $x^2 \neq x^*x$, so $x^*x = xx^*$ and $\text{char } R = 4$.

Let x and y be any two elements of $G \setminus N$. The elements $x + x^*$ and $y + y^*$ are in \mathcal{S}_4 , so they anticommute. The equation $(x + x^*)(y + y^*) = -(y + y^*)(x + x^*)$ gives

$$(2.3) \quad xy + xy^* + x^*y + x^*y^* + yx + yx^* + y^*x + y^*x^* = 0.$$

Now $xy \in N$, so $y^*x^* = (xy)^* = xy$. Similarly, $xy^* = yx^*$, $x^*y = y^*x$, $x^*y^* = yx$ and (2.3) becomes $2(xy + xy^* + x^*y + yx) = 0$. Since $xy \neq xy^*$ and $xy \neq x^*y$, the only possibility is $xy = yx$; thus the elements of $G \setminus N$ commute.

Let $n \in N$ and $x \notin N$. Then $nx \notin N$ so $(nx)x = x(nx)$, implying $nx = xn$. Since N is abelian, it follows that G is abelian and we have the first situation described in the theorem.

2) Assume $\text{char } R = 4$, N is abelian and there exists $s \in N$ with $s^2 = 1$ and $n^* = n$ or sn for all $n \in N$. We may assume that $n^* \neq n$ for some $n \in N$, for otherwise we are in Case 1 which has just been settled. In particular, we may assume that $s \neq 1$ and then, since $s^* = s$ or $s^* = ss = 1$, which is false, we get $s^* = s$.

Let $x \notin N$. Then $x^* \neq x$ and $x + x^* \in \mathcal{S}_4$ anticommutes with itself giving $x^2 = (x^*)^2 = (x^2)^*$, $xx^* = x^*x$, as before. Furthermore, $sx \notin N$, so $x + x^*$ and $(sx) + (sx)^* = sx + x^*s$ anticommute, that is,

$$\begin{aligned} 0 &= (sx + x^*s)(x + x^*) + (x + x^*)(sx + x^*s) \\ &= sx^2 + sxx^* + x^*sx + x^*sx^* + xsx + xx^*s + x^*sx + (x^*)^2s \\ &= 2sx^2 + 2sxx^* + 2x^*sx + x^*sx^* + xsx, \end{aligned}$$

remembering that N is abelian and noting that xx^* and x^2 are in N . Now $sx^2 \neq sxx^*$ because $x \neq x^*$ and $sx^2 \neq x^*sx$ because $\mathcal{S}_3 = \emptyset$ means $sx \neq (sx)^* = x^*s$. In characteristic 4, it follows that $sx^2 = x^*sx^* = xsx$, so $sx = xs$. Thus s is central in G .

Let n be any element of N with $n^* = n$. So $2n \in \mathcal{S}_1$. Let x be an element of G not in N . Then $x + x^* \in \mathcal{S}_4$, so $2n$ and $x + x^*$ anticommute, giving

$$2nx + 2nx^* + 2xn + 2x^*n = 0.$$

Now $nx \neq nx^*$ and $nx \neq (nx)^* = x^*n$, so $nx = xn$. This shows that n commutes with all elements not in N . Since n also commutes with elements in N (N is abelian), this element is central in G . In particular, for any $x \notin N$, both xx^* and x^2 are central.

Suppose $n \in N$ and $n^* = sn$. Let x be an element not in N . Then nx is not in N (so $nx \neq (nx)^* = sx^*n$) and $nx + (nx)^* = nx + sx^*n$ anticommutes with $x + x^*$ giving

$$nx^2 + nxx^* + sx^*nx + sx^*nx^* + xnx + sxx^*n + x^*nx + s(x^*)^2n = 0.$$

In characteristic 4, two sets of four elements on the left must consist of equal elements. Now $nx^2 \neq nxx^*$ and $nx^2 \neq sx^*nx$ (because $nx \neq (nx)^*$), so three elements of $\{sx^*nx^*, xnx, sxx^*n, x^*nx, s(x^*)^2n = sx^2n\}$ are equal and equal to nx^2 . Since $xnx \neq x^*nx$ and $sxx^*n \neq sx^2n$, there are four possibilities:

- i. $nx^2 = sx^*nx^* = xnx = sxx^*n$,
- ii. $nx^2 = sx^*nx^* = xnx = sx^2n$,
- iii. $nx^2 = sx^*nx^* = x^*nx = sxx^*n$,
- iv. $nx^2 = sx^*nx^* = x^*nx = sx^2n$.

We have previously observed that x and x^* commute, so $xx^*n = x^*xn$. Thus Case i implies both $nx = xn$ and $nx^* = xn$, so $x^* = x$, which is wrong. Case ii says $nx^2 = sx^2n = snx^2$ (squares are central), an obvious contradiction, and Case iv gives the same contradiction, so we must be in Case iii. Here $x^2n = nx^2 = sxx^*n$ gives $x = sx^*$, hence $x^* = sx$, and then $nx^2 = x^*nx = sxxn$ gives $nx = sxn$. As a biproduct of these arguments, we have learned also that $n^* \neq n$ implies n is not central, so

$$\mathcal{Z}(G) = \{n \in N \mid n^* = n\}$$

and, furthermore, the commutator of an element of N with an element not in N is 1 or s .

Fix an $a \notin N$ and, with $n, m \in N$, use $G = N \cup Na$ to find the commutator of two elements $x = na$, $y = ma$, neither of which is in N . On the one hand

$$xy = (na)(ma) = \begin{cases} nma^2 & \text{if } am = ma \\ snma^2 & \text{if } am = sma \end{cases}$$

while, similarly, $yx = mna^2$ or $smna^2$. Since $(m, n) = 1$ or s , so also (x, y) is 1 or s .

Our reasonings have shown that $G' = \{1, s\}$ and that

$$g^* = \begin{cases} g & \text{if } g \in \mathcal{Z}(G) \\ sg & \text{if } g \notin \mathcal{Z}(G). \end{cases}$$

Since $*$ is an involution, G must have the LC property, for this reason: if $gh = hg$ with none of g, h, gh central, then $sgh = (gh)^* = h^*g^* = (sh)(sg) = hg = gh$, which cannot be. Thus we are in the second situation described by the theorem.

3) Finally, assume that $\text{char } R = 4$ and that N is a nonabelian group with a unique nonidentity commutator, s (necessarily central in N and of order 2), and $n^* = n$ or sn for every $n \in N$. As an involution on a nonabelian group, $*$ cannot be the identity on N , so $s \neq 1$ and, as in the previous case, we have $s^* = s$. Moreover, as in previous cases, we have $x^2 = (x^2)^*$ and $xx^* = x^*x$ for any $x \notin N$.

Let $x \notin N$. Then also $sx \notin N$ so $x + x^*$ and $(sx) + (sx)^* = sx + x^*s$ anticommute, that is,

$$\begin{aligned} 0 &= (sx + x^*s)(x + x^*) + (x + x^*)(sx + x^*s) \\ &= sx^2 + sxx^* + x^*sx + x^*sx^* + xsx + xx^*s + x^*sx + (x^*)^2s \\ &= 2sx^2 + 2sxx^* + 2x^*sx + x^*sx^* + xsx \end{aligned}$$

using the facts that xx^* and $(x^*)^2 = x^2$ are in N and hence commute with s , which is central in N . Now $sx^2 \neq sxx^*$ because $x \neq x^*$ and $sx^2 \neq x^*sx$ because $\mathcal{S}_3 = \emptyset$ means $sx \neq (sx)^* = x^*s$. In characteristic 4, it follows that $sx^2 = x^*sx^* = xsx$, so $sx = xs$. Thus, not only is s central in N , but it is actually central in G .

Let n be any element of N satisfying $n^* = n$. Then $2n \in \mathcal{S}_1$. Let x be an element of G not in N . Then $x + x^* \in \mathcal{S}_4$, so $2n$ and $x + x^*$ anticommute, that is,

$$0 = 2nx + 2nx^* + 2xn + 2x^*n.$$

Now $nx \neq nx^*$ and $nx \neq (nx)^* = x^*n$ (remember that $\mathcal{S}_3 = \emptyset$), so $nx = xn$. This shows that n commutes with all elements of $G \setminus N$. Take any m in N and $x \notin N$. Then n and mx commute, so $nm x = mnx = mnx$ so $mn = nm$. This proves that if $n^* = n \in N$, then n is central in G . In particular, for any $x \notin N$, we have both xx^* and x^2 central in G .

Since $*$ is not the identity on N , there exists $n \in N$ with $n^* = sn$. Let x be an element of G not in N . Then nx is not in N (so $nx \neq (nx)^* = sx^*n$) and $nx + (nx)^* = nx + sx^*n$ anticommutes with $x + x^*$ giving

$$0 = nx^2 + nxx^* + sx^*nx + sx^*nx^* + xnx + sxx^*n + x^*nx + s(x^*)^2n.$$

Now $nx^2 \neq nxx^*$ and $nx^2 \neq sx^*nx$ (because $nx \neq (nx)^*$), so three elements of $\{sx^*nx^*, xnx, sxx^*n, x^*nx, s(x^*)^2n = sx^2n\}$ are equal and equal to nx^2 . Now $xnx \neq x^*nx$ and $sxx^*n \neq sx^2n$, so there are apparently four possibilities:

- i. $nx^2 = sx^*nx^* = xnx = sxx^*n$,
- ii. $nx^2 = sx^*nx^* = xnx = sx^2n$,
- iii. $nx^2 = sx^*nx^* = x^*nx = sxx^*n$,
- iv. $nx^2 = sx^*nx^* = x^*nx = sx^2n$.

Case i implies both $nx = xn$ and $nx^* = xn$ (using $sxx^*n = sx^*xn$) and hence $x = x^*$, a contradiction. Cases ii and iv say $nx^2 = sx^2n = snx^2$ by centrality of x^2 , an obvious contradiction, so the situation is as described in Case iii. So $x^2n = nx^2 = sxx^*n$ and hence $x = sx^*$ and $x^* = sx$, and then $nx^2 = x^*nx = sxnx$, so $nx = sxn$. In particular, we learn that $n^* \neq n$ implies n is not central, so, just as before,

$$\mathcal{Z}(G) = \{n \in N \mid n^* = n\}.$$

We have also seen that any commutator (n, x) , with $n \in N$ and $x \notin N$, is 1 or s . The commutator of any two elements of N is also 1 or s . Fix an $a \notin N$, take $n, m \in N$ and use $G = N \cup Na$ to find the commutator of two elements $x = na$, $y = ma$ neither of which is in N . On the one hand

$$xy = (na)(ma) = \begin{cases} nma^2 & \text{if } am = ma \\ snma^2 & \text{if } am = sma \end{cases}$$

while, similarly, $yx = mna^2$ or $smna^2$. Since the commutator $(n, m) = 1$ or s , it is clear that (x, y) is also 1 or s . All this shows that $G' = \{1, s\}$ and

the involution $*$ is defined by

$$g^* = \begin{cases} g & \text{if } g \in \mathcal{Z}(G) \\ sg & \text{if } g \notin \mathcal{Z}(G). \end{cases}$$

It remains just to show that G has the limited commutativity (LC) property (so that G is SLC).

Suppose two elements $x \notin N$, $y \notin N$ commute. Then $xy = yx$ implies $xyx^* = yxx^* = xx^*y$, $yx^* = x^*y$ and, similarly, $xy^* = y^*x$. Now $x + x^*$ and $y + y^*$ are in \mathcal{S}_4 , so they anticommute, giving

$$\begin{aligned} 0 &= (x + x^*)(y + y^*) + (y + y^*)(x + x^*) \\ &= 2xy + 2x^*y^* + 2xy^* + 2x^*y. \end{aligned}$$

The only possibility is $xy = x^*y^*$ and $xy^* = x^*y$. The former equation says $xy = (yx)^* = (xy)^*$ so xy is central.

We have seen that $nx = xn$, $n \in N$, $x \notin N$, implies that n is not central. Finally, if $n, m \in N$ commute and neither is central, then $n^* = sn$, $m^* = sm$ and so $(nm)^* = m^*n^* = (sm)(sn) = mn = nm$, that is, nm is central. Thus G indeed has the LC property, the situation is as described in (2), and the proof is complete. \square

3. SYMMETRIC ELEMENTS COMMUTE

In this section, we find conditions under which the \sharp -symmetric elements of a group ring RG commute and thus determine when $(RG)^+$ is a subring of RG . As mentioned, this problem has been previously studied [CM06] although, unfortunately, the possible existence of 2-torsion elements in R was overlooked. The correct result is this.

Theorem 3.1. *Let $g \mapsto g^*$ denote an involution on a group G and let $\sigma: G \mapsto \{\pm 1\}$ be an orientation homomorphism which is not identically 1. Set $N = \ker \sigma$. Let R be a commutative ring with 1 and let $R_2 = \{r \in R \mid 2r = 0\}$. For $\alpha = \sum_{g \in G} \alpha_g g$ in the group ring RG , define $\alpha^\sharp = \sum_{g \in G} \sigma(g) \alpha_g g^*$. Assume $*$ and σ are compatible in the sense that $\sigma(g) = \sigma(g^*)$ for all $g \in G$. If the set $(RG)^+$ of \sharp -symmetric elements of RG is commutative, then G is abelian or else one of the following situations arises:*

- (1) $R_2^2 = \{0\}$, N is abelian, $x^* = x$ for $x \in G \setminus N$ and $n^* = a^{-1}na$ for any $n \in N$ and any $a \in G \setminus N$;
- (2) the characteristic of R is 4, N is abelian, G has a unique nonidentity commutator, s , $x^* = sx$ for $x \in G \setminus N$, and $n^* = a^{-1}na$ for any $n \in N$ and any $a \in G \setminus N$;
- (3) $R_2^2 = \{0\}$, N is SLC with unique nonidentity commutator s and canonical involution $*$, and $G = N\langle a \rangle$ is the product of N and a central subgroup generated by an element a satisfying $a^* = sa$.
- (4) the characteristic of R is 4 and both N and G are SLC groups with canonical involution.

Conversely, suppose that G is a group with an index 2 subgroup N and $\sigma: G \rightarrow \{\pm 1\}$ is the orientation homomorphism with kernel N . Furthermore, suppose that either G is abelian with $*$ the identity map or the conditions of (1), (2), (3) or (4) are satisfied. Then $*$ is an involution on G and the set $(RG)^+$ of \sharp -symmetric elements is a commutative set.

Proof. As before, we begin by determining the nature of the elements of $(RG)^+$. With the S_i as in Section 2 and $\alpha \in RG$ written in the form (2.1), we have

$$\alpha^\sharp = \alpha \text{ if and only if } \begin{cases} \alpha_g = \alpha_{g*} & \text{if } g \in S_2 \\ \alpha_g = -\alpha_g & \text{if } g \in S_3 \\ \alpha_g = -\alpha_g^* & \text{if } g \in S_4. \end{cases}$$

Thus $(RG)^+$ is spanned over R by the union of the sets

$$\begin{aligned} \mathcal{T}_1 &= \{n \in N \mid n^* = n\}, \\ \mathcal{T}_2 &= \{n + n^* \mid n \in N, n^* \neq n\}, \\ \mathcal{T}_3 &= \{\alpha_x x \mid x^* = x \notin N, 2\alpha_x = 0\}, \\ \mathcal{T}_4 &= \{x - x^* \mid x \notin N, x^* \neq x\}. \end{aligned}$$

Clearly, the elements of $(RG)^+$ commute if and only if any two elements from the union of the \mathcal{T}_i commute.

We begin our proof with the converse, noting initially that if G is abelian, then RG is a commutative ring, so the elements of $(RG)^+$ commute.

(1) Suppose $R_2^2 = \{0\}$, N is abelian, $x^* = x$ for $x \notin N$ (so $\mathcal{T}_4 = \emptyset$) and $n^* = a^{-1}na$ for any $n \in N$ and any $a \notin N$. We first note that n^* is well-defined because if a and b are any two elements of $G \setminus N$, then $b = ma$ for some $m \in N$ because N has index 2 in G , so $b^{-1}nb = a^{-1}m^{-1}nma = a^{-1}na$ because N is abelian. Now fix $a \notin N$. To show that $*$ is an involution, there are three cases to consider.

- If $n_1, n_2 \in N$, then $(n_1 n_2)^* = a^{-1}n_1 n_2 a = (a^{-1}n_1 a)(a^{-1}n_2 a) = (a^{-1}n_2 a)(a^{-1}n_1 a)$ (because N is abelian), and this is $n_2^* n_1^*$.
- If $n_1 \in N$ and $x \notin N$, then $x = n_2 a$ for some $n_2 \in N$. Since $n_1 x \notin N$, $(n_1 x)^* = n_1 x$, while $x^* n_1^* = x a^{-1} n_1 a = n_2 a a^{-1} n_1 a = n_2 n_1 a = n_1 n_2 a = n_1 x$ also.
- If $x \notin N$ and $y \notin N$, then $x = a n_1$, $y = a n_2$ for some $n_1, n_2 \in N$. Since $xy \in N$, $(xy)^* = a^{-1}xya = a^{-1}a n_1 a n_2 a = n_1 a n_2 a$, whereas $y^* x^* = yx = a n_2 a n_1$. Now $a n_2 \notin N$ and $a \notin N$, so their product $a n_2 a$ is in N and so commutes with n_1 . This gives $y^* x^* = n_1 a n_2 a = (xy)^*$ in this case as well.

To show that the elements of $(RG)^+$ commute, it is sufficient to show that any two elements of $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ commute ($\mathcal{T}_4 = \emptyset$). Any two elements of $\mathcal{T}_1 \cup \mathcal{T}_2$ commute because N is abelian. Elements $\alpha_x x$ and $\alpha_y y$ of \mathcal{T}_3 commute because, as elements of R_2 , α_x and α_y have product 0. Let $n \in \mathcal{T}_1$ and $\alpha_x x \in \mathcal{T}_3$. Since the restriction of $*$ to N is conjugation by any element

not in N , $n = n^* = x^{-1}nx$, so $xn = nx$ implying that n and $\alpha_x x$ commute. Finally, if $n+n^* \in \mathcal{T}_2$ and $\alpha_x x \in \mathcal{T}_3$, then $n^* = x^{-1}nx$, so $x(n+n^*) = xn+nx$, whereas $(n+n^*)x = nx + x^{-1}nx^2 = nx + xn$ too (because x^2 is in N and so commutes with n).

(2) Suppose $\text{char } R = 4$, N is abelian, G has a unique nonidentity commutator, s (necessarily central and of order 2), $x^* = sx$ for $x \notin N$ and $n^* = a^{-1}na$ for any $n \in N$ and any $a \notin N$. As in our discussion of (1) just above, n^* is well-defined and, to show that $*$ is an involution on G , there are three cases to consider.

- If $n_1, n_2 \in N$, then $(n_1 n_2)^* = n_2^* n_1^*$ as before.
- If $n_1 \in N$ and $x \notin N$, then $x = n_2 a$ for some $n_2 \in N$. Since $n_1 x \notin N$, $(n_1 x)^* = s n_1 x$, while $x^* n_1^* = s x a^{-1} n_1 a = s n_2 a a^{-1} n_1 a = s n_2 n_1 a = s n_1 n_2 a = s n_1 x$ also.
- If $x \notin N$ and $y \notin N$, then $x = a n_1$, $y = a n_2$ for some $n_1, n_2 \in N$. Since $xy \in N$, $(xy)^* = a^{-1} x y a = a^{-1} a n_1 a n_2 a = n_1 a n_2 a$, whereas $y^* x^* = (s y)(s x) = y x = a n_2 a n_1$. As above, the elements $(xy)^*$ and $y^* x^*$ are the same.

This time, $\mathcal{T}_3 = \emptyset$, so to show that the elements of $(RG)^+$ commute, it suffices to show that any two elements of $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_4$ commute. Any two elements of $\mathcal{T}_1 \cup \mathcal{T}_2$ commute because N is abelian. Let $x - x^* = x - sx = (1-s)x$ and $y - y^* = (1-s)y$ be two elements of \mathcal{T}_4 . Then $(x - x^*)(y - y^*) = (1-s)^2 xy = 2(1-s)xy$, whereas

$$(y - y^*)(x - x^*) = 2(1-s)yx = \begin{cases} 2(1-s)xy & \text{if } yx = xy \\ 2(1-s)sxy & \text{if } yx \neq xy. \end{cases}$$

Since $2(1-s)s = 2(s-1) = -2(1-s) = 2(1-s)$ (in characteristic 4), we see that $x - x^*$ and $y - y^*$ commute in either case. Let $n \in \mathcal{T}_1$ and $x - x^* \in \mathcal{T}_4$. Since $n = n^* = x^{-1}nx$, n and x commute, so n and $x - x^* = (1-s)x$ commute. Finally, let $n + n^* \in \mathcal{T}_2$ and $x - x^* = (1-s)x \in \mathcal{T}_4$. If $nx = xn$, then $(nx)^* = (xn)^*$, that is, $x^* n^* = n^* x^*$ which says $sxn^* = sn^*x$, so $n^*x = xn^*$ too. Then $(n + n^*)(x - x^*) = (1-s)(nx + n^*x)$, whereas $(x - x^*)(n + n^*) = (1-s)(xn + xn^*)$ and these elements are the same. If $nx \neq xn$, then $nx = sxn$, $n^* = x^{-1}nx = sn$, $n + n^* = (1+s)n$ and both products $(n + n^*)(x - x^*)$ and $(x - x^*)(n + n^*)$ are 0 because $(1+s)(1-s) = 0$. In any case, an element of \mathcal{T}_2 always commutes with an element of \mathcal{T}_4 .

(3) Suppose $R_2^2 = \{0\}$, N is SLC with unique nonidentity commutator s and canonical involution $*$, and $G = N\langle a \rangle$ is the product of N and a cyclic subgroup $\langle a \rangle$ generated by a central element $a \notin N$ satisfying $a^* = sa$. Since $G = N \cup Na$, if $x \notin N$, there exists $n \in N$ with $x = na$, so $x^* = a^* n^* = sn^* a$ (because a is central). Thus $*$ extends to a map $G \rightarrow G$ which is an involution on G because $*$ is an involution on N ,

- $[n_1(n_2 a)]^* = [(n_1 n_2) a]^* = s(n_1 n_2)^* a = s n_2^* n_1^* a = s n_2^* a n_1^*$ (a is central) $= (n_2 a)^* n_1^*$,
and

- $[(n_1a)(n_2a)]^* = [(n_1n_2)a^2]^* = (a^2)^*(n_1n_2)^* = (a^*)^2(n_1n_2)^*$
 $= (sa)^2(n_1n_2)^* = (n_1n_2)^*a^2$ because $s^2 = 1$,
while $(n_2a)^*(n_1a)^* = sn_2^*asn_1^*a = (n_1n_2)^*a^2$ too.

To show that the elements of $(RG)^+$ commute, we first remark that elements $n \in N$ with $n^* = n$ and of the form $n + n^*$, $n \in N$, are known to be central in the group ring RN because N is SLC [GJM96, Corollary III.4.3]. It follows that any two elements of $\mathcal{T}_1 \cup \mathcal{T}_2$ commute. Elements $\alpha_x x$ and $\alpha_y y$ in \mathcal{T}_3 commute because $\alpha_x \alpha_y = 0$. If $x - x^*$ is in \mathcal{T}_4 , then $x = na$ for some $n \in N$, so $x - x^* = na - sn^*a = (n - sn^*)a \in (RN)a$. Now consider two elements $(n - sn^*)a$ and $(m - sm^*)a$ of this form. Commutativity is equivalent to $(n - sn^*)(m - sm^*) = (m - sm^*)(n - sn^*)$, that is,

$$nm - snm^* - sn^*m + n^*m^* = mn - smn^* - sm^*n + m^*n^*.$$

If n or m is central, say n , then $n^* = n$ and this equation is easily seen to be satisfied. On the other hand, if neither n nor m is central, then $n^* = sn$, $m^* = sm$ and each side of the equation is 0. In any case, two elements of \mathcal{T}_4 commute. Commutativity of all other pairs of elements in $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4$ follows from centrality of a .

(4) Suppose $\text{char } R = 4$ and both N and G are SLC groups with canonical involution. To show that the elements of $(RG)^+$ commute, it suffices to show that any two elements of $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4$ commute. In an SLC group G , $*$ -symmetric elements and elements of the form $g + g^*$ are central [GJM96, Corollary III.4.3], so the elements of $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ are central. It remains then only to consider two elements of \mathcal{T}_4 and these commute with precisely the argument given above in case (2).

We now attack the first statement of the theorem, assuming that G is nonabelian and $\alpha \mapsto \alpha^\#$ is a nontrivial oriented group involution on a group ring RG relative to which the symmetric elements commute. Thus $N = \ker \sigma$ has index 2 in G and, since σ is not identically 1, we must have $\text{char } R \neq 2$. On the other hand, $\sigma|_N \equiv 1$ and the restriction of $*$ to N is an involution $N \rightarrow N$ (by compatibility), so N is abelian or an SLC group [JM06, Theorem 2.4].

Assume that N is abelian.

Let a be any element of $G \setminus N$. If the elements of $G \setminus N$ commute, then a and na commute for any $n \in N$, so a and n commute and it is clear that G is abelian, a contradiction. Thus we can always assume the existence of elements $x, y \in G \setminus N$ with $xy \neq yx$.

Suppose that $x^* = x$ for all $x \notin N$. Let $n \in N$ and $a \notin N$. Then $G = N \cup Na$ and $na \notin N$, so $na = (na)^* = a^*n^* = an^*$ giving $n^* = a^{-1}na$. Let $\alpha_x, \alpha_y \in R_2$ and let x and y be elements of $G \setminus N$ that do not commute. Since $x^* = x$ and $y^* = y$, the elements $\alpha_x x$, $\alpha_y y$ are in \mathcal{T}_3 , so they commute, implying $0 = \alpha_x \alpha_y (xy - yx)$. So $\alpha_x \alpha_y = 0$ and we have the situation

described in (1). Henceforth, then, we assume the existence of an element $x \notin N$ with $x^* \neq x$.

Let $n \in N$ satisfy $n^* = n$. Let $x \notin N$ be such that $x^* \neq x$. Then $x - x^* \in \mathcal{T}_4$ and this element commutes with n (because $n \in \mathcal{T}_1$). The equation $n(x - x^*) = (x - x^*)n$ says $nx - nx^* = xn - x^*n$, so $nx + x^*n = xn + nx^*$. Each side of this equation is the sum of group elements. In characteristic different from 2, the only possibility is $xn = nx$.

Let $n \in N$ with $n^* \neq n$ and again take $x \notin N$ with $x^* \neq x$. The elements $n + n^* \in \mathcal{T}_2$ and $x - x^* \in \mathcal{T}_4$ commute, so $nx - nx^* + n^*x - n^*x^* = xn + xn^* - x^*n - x^*n^*$, that is,

$$nx + n^*x + x^*n + x^*n^* = xn + xn^* + nx^* + n^*x^*.$$

Since $x \neq x^*$ and $n \neq n^*$ (and $\text{char } R \neq 2$), nx appears on the left side of this equation (with nonzero coefficient), so it must be on the right and there are apparently three possibilities— $nx = xn$, $nx = xn^*$ or $nx = n^*x^*$ —but, in fact, just two because we shall argue that the third alternative implies one of the first two. For this, assume $nx = n^*x^* = (xn)^*$. We will show that this implies $nx = xn$ or $nx = xn^*$. The element $(nx) - (nx)^* = nx - xn$ is in \mathcal{T}_4 , so it commutes with $n + n^*$, that is,

$$n xn + n x n^* - x n^2 - x n n^* = n^2 x - n x n + n^* n x - n^* x n,$$

which, rewritten, gives

$$2n xn + n x n^* + n^* x n = n^2 x + n^* n x + x n^2 + x n n^*.$$

Since $n xn$ appears on the left side of this equation with coefficient 2, two terms on the right side are equal, and equal to $n xn$. Since $n^* \neq n$, $x n^2 \neq x n n^*$ and $n^2 x \neq n n^* x = n^* n x$, the choices are

- i. $n xn = n^2 x = x n^2$;
- ii. $n xn = n^2 x = x n n^*$;
- iii. $n xn = n^* n x = x n^2$;
- iv. $n xn = n^* n x = x n n^*$.

In cases i, ii and iii, $nx = xn$, and in case iv, since n and n^* commute in an SLC group, we have $n xn = x n n^* = x n^* n$ and so $nx = x n^*$. Thus $nx = n^* x^*$ implies $nx = xn$ or $nx = x n^*$ as claimed.

At this point, we have shown that for any $n \in N$ (whether or not $n^* = n$) and for any $x \notin N$ with $x^* \neq x$, either $nx = xn$ or $n^* = x^{-1}nx$.

Now fix $a \notin N$ with $a^* \neq a$. Let

$$H = \{n \in N \mid an = na\} \text{ and } K = \{n \in N \mid n^* = a^{-1}na\}.$$

Each of these sets is a subgroup of N (because N is abelian) and $N = H \cup K$, so either $H \subseteq K = N$ or $K \subseteq H = N$. In the second case, $G = N \cup Na$ is abelian, contrary to assumption.

Thus we have the first case, $K = N$ and $n^* = a^{-1}na$ for all $n \in N$. Let $x \notin N$. Then $x = na$ for some $n \in N$, so $x^* = a^*n^* = a^*a^{-1}na = sx$ with $s = a^*a^{-1} \neq 1$ independent of x . In particular, $x = (x^*)^* = sx^* = s^2x$, so

$s^2 = 1$. Now $s \in N$, so $s^* = a^{-1}sa = a^{-1}a^*$ and $s = (s^*)^* = a(a^{-1})^* = asa^{-1}$ (because $a^{-1} \notin N$). It follows that $a^{-1}s = sa^{-1}$, so s and a commute and $s^* = a^{-1}sa = s$. As well, for any $n \in N$, s and n commute (because s is in the abelian group N), so $s(na) = (na)s$. It follows that s commutes with any $x \notin N$, so s is central in G .

Now $G \setminus N = Na$ is not commutative (else G is abelian), so there exist elements $x - x^*, y - y^* \in \mathcal{T}_4$ with $xy \neq yx$. Thus $x - x^* = x - sx = (1 - s)x$ and $y - y^* = (1 - s)y$ commute, so $(1 - s)^2xy = (1 - s)^2yx$, that is, $2(1 - s)xy = 2(1 - s)yx$, equivalently,

$$2xy + 2syz = 2yx + 2sxy.$$

Since $xy \neq yx$, the only possibility is $\text{char } R = 4$, $xy = syx$ and $yx = sxy$. So the commutator $(x, y) = x^{-1}y^{-1}xy = (yx)^{-1}(xy) = s$. Let $n_1 \in N$ and $x \notin N$. Then $x = n_2a$ for some $n_2 \in N$ and the commutator $(n_1, x) = n_1^{-1}a^{-1}n_2^{-1}n_1n_2a = n_1^{-1}a^{-1}n_1a$ because N is abelian. Neither a^{-1} nor n_1a are in N , so the commutator of these two elements is 1 or s ; that is, $a^{-1}n_1a = n_1aa^{-1} = n_1$ or $a^{-1}n_1a = sn_1aa^{-1} = sn_1$, giving $(n_1, x) = n_1^{-1}n_1 = 1$ or $(n_1, x) = n_1^{-1}sn_1 = s$. It follows that $G' = \{1, s\}$ and we have the situation described by case (2) of the theorem.

Assume that N is an SLC group (with unique commutator s).

We first prove that the restriction of $*$ to N is the canonical involution on N —see (1.1). Let $n \in N$ with $n^* \neq n$ and suppose that n is not central, selecting $m \in M$ with $nm \neq mn$ (so $nm = smn$). Then n and $m + m^*$ commute, giving $nm + nm^* = mn + m^*n$ and hence $nm = m^*n$. So $m^* = nm n^{-1} = sm$. Now n and mn do not commute and so, similarly, $(mn)^* = smn$. But $(mn)^* = n^*m^* = snm$, so $nm = mn$, a contradiction. This shows that if $n = n^*$, then n must be central.

Suppose $n \in N$ is not central and choose $m \in N$ with $nm \neq mn$. Then m is also noncentral, so both $n^* \neq n$ and $m^* \neq m$. The elements $n + n^*$ and $m + m^*$ are both in \mathcal{T}_2 , so they commute, giving

$$(3.1) \quad nm + nm^* + n^*m + n^*m^* = mn + mn^* + m^*n + m^*n^*.$$

Since $nm \neq nm^*$ and $n^*m \neq n^*m^*$, the element nm appears on the left side, so it appears on the right. Now $nm \neq mn$ and $nm \neq m^*n^*$, else $nm = (nm)^*$ would imply that nm is central and n and m commute. The possibilities are, therefore, $nm = mn^*$ (implying $n^* = m^{-1}nm = sn$) or $nm = m^*n$ (implying $m^* = nm n^{-1} = sm$). Suppose $n^* = sn$ and apply what we have just learned to the noncentral elements m and nm . Either $m^* = sm$ or $(nm)^* = snm$, the latter giving $m^*n^* = snm$, so $sm^*n = snm = mn$ and $m^* = sm$. Similarly, if $m^* = sm$, we get $n^* = sn$ too. Thus $m^* = sm$ for any noncentral element m .

Let n be central and m noncentral. Then nm is noncentral, so $(nm)^* = snm$. This gives $m^*n^* = snm$, $smn^* = snm = smn$, so $n^* = n$. We have shown that n is central if and only if $n^* = n$ and hence m is noncentral if and only if $m^* = sm$. Indeed, the restriction of $*$ to N is canonical.

Suppose next that $x^* = x$ for all $x \notin N$. Then for any $n \in N$ and any $a \notin N$, $na \notin N$, so $na = (na)^* = a^*n^* = an^*$ giving $n^* = a^{-1}na$. In particular, $a^{-1}sa = s^* = s$, so s is central in G . Take $n \in N$ not central. Then $n^* = sn$, so $na = san$ (and $an = sna$). Choose $m \in N$ with $nm \neq mn$. Since $(nm)a \notin N$, $[n(ma)]^* = [(nm)a]^* = nma$, while, since $ma \notin N$, $(ma)^*n^* = (ma)sn = sman = sm(sna) = mna$. These calculations show $[(n(ma))]^* \neq (ma)^*n^*$, a contradiction. Thus there exists $a \notin N$ with $a^* \neq a$.

Now $s^* = s$, so $s \in \mathcal{T}_1$. This element commutes with $a - a^* \in \mathcal{T}_4$, so $sa - sa^* = as - a^*s$, $sa + a^*s = as + sa^*$ giving $sa = as$ (remember that $\text{char } R \neq 2$). Moreover, for any $x \notin N$, $x = na$ for some $n \in N$, and $sx = s(na) = nsa = (na)s = xs$, so s is central in G .

Let n be any element of N . We proceed exactly as in the N abelian case.

If $n^* = n$, then n commutes with $a - a^*$, so $na = an$. For example, aa^* is invariant under $*$, so $(aa^*)a = a(aa^*)$ implying $aa^* = a^*a$.

If $n^* \neq n$, the elements $n + n^*$ and $a - a^*$ commute, so

$$na - na^* + n^*a - n^*a^* = an + an^* - a^*n - a^*n^*,$$

which, rewritten, says,

$$na + n^*a + a^*n + a^*n^* = an + an^* + na^* + n^*a^*,$$

giving rise apparently to three possibilities,

$$(3.2) \quad na = an, \quad na = an^*, \quad na = n^*a^*.$$

Just as before, we argue that the third alternative implies one of the first two. So assume $na = n^*a^*$. Then $n \neq n^*$ means $n^* = sn$ and $na = n^*a^* = sna^*$ gives $a^* = sa$ too. Let $b = an$. If $b = b^*$, then $an = n^*a^* = (sn)(sa) = na$. If $b^* \neq b$, then just as with a , either $nb = bn$ or $nb = bn^*$ or $nb = n^*b^*$. If $nb = bn$, then $nan = an^2$, so $na = an$. If $nb = bn^*$, then $nan = ann^* = an^*n$, so $na = an^*$. If $nb = n^*b^*$, then $nan = n^*n^*a^* = (sn)n^*(sa) = nn^*a$, so $an = n^*a$ and, applying the involution $*$, $n^*a^* = a^*n$; that is $(sn)(sa) = (sa)n$, so $na = a(sn) = an^*$. All this shows that for any $n \in N$ (whether or not $n^* = n$), either $na = an$ or $na = an^*$.

We distinguish two cases.

Case I. Suppose $na = an$ for all $n \in N$. Then it is easy to see that a is central, $G = N\langle a \rangle$ is product of the groups N and $\langle a \rangle$ and so s is in fact a unique nonidentity commutator in G . We claim we may assume $a^* = sa$, without loss of generality.

There exist $n, m \in N$ with $nm \neq mn$. This implies $n^* \neq n$ and $m^* \neq m$, so $n^* = sn$ and $m^* = sm$. Let $x = na$, $y = ma$ and note that $xy = (nm)a^2$ while $yx = (mn)a^2$, so $xy \neq yx$.

We now argue, by way of contradiction, that either $x^* = x$ or $y^* = y$. Thus we suppose that $x^* \neq x$ and $y^* \neq y$ and conclude that the \sharp -symmetric elements $x - x^*$ and $y - y^*$ must commute, giving

$$xy - xy^* - x^*y + x^*y^* = yx - yx^* - y^*x + y^*x^*;$$

that is,

$$(3.3) \quad xy + x^*y^* + yx^* + y^*x = yx + y^*x^* + xy^* + x^*y.$$

Suppose $x^*y = yx^*$. Then $x^*yy^* = yx^*y^*$. Since yy^* is central in N , it is central in G , so $yy^*x^* = yx^*y^*$, giving $y^*x^* = x^*y^*$ and so $xy = yx$, which is not true. So $x^*y \neq yx^*$, hence $yx^* = sx^*y$ and, similarly, $y^*x = sxy^*$. Now (3.3) reads

$$(3.4) \quad xy + x^*y^* + sx^*y + sxy^* = yx + y^*x^* + xy^* + x^*y.$$

If $x^* = sx$, then $a^*n^* = sna$, so $sa^*n = san$ and $a^* = a$, a contradiction. Thus $x^* \neq sx$ and, for the same reason, $y^* \neq sy$. It follows that the three elements x^*y^*, sx^*y, sxy^* are distinct and so xy appears on the left side of (3.4) and hence on the right.

The only possibility is $xy = y^*x^* = (xy)^*$; that is xy is a $*$ -symmetric element of N . This means xy is central in N , so xy is central in $N\langle a \rangle = G$ implying $xy = yx$, a contradiction. It follows, as claimed, that either $x^* = x$ or $y^* = y$.

The first possibility says $a^*(sn) = na = an$, so $a^* = sa$, and we reach the same conclusion if $y^* = y$. Thus $a^* = sa$.

Finally, let $n, m \in N$ with $nm \neq mn$. Then neither n nor m is central in the SLC group N , so $n^* = sn$ and $m^* = sm$. Let $x = na$ and note that $x^* = sn^*a = na = x$ and, similarly, $y^* = y$. Note too that $xy \neq yx$. Let $\alpha_x, \alpha_y \in R_2$. Then $\alpha_x x$ and $\alpha_y y$ are in \mathcal{T}_3 . So they must commute, giving $\alpha_x \alpha_y (xy - yx) = 0$. Since $xy \neq yx$, it follows that $\alpha_x \alpha_y = 0$. Thus $R_2^2 = \{0\}$ and the situation is as described in (3).

Case II. Suppose that for each $a \notin N$ with $a^* \neq a$, we have $an \neq na$ for some $n \in N$. This implies that if $a \notin N$ and $a = a^*$, then $na = an$ for all $n \in N$ and, since $G = N \cup Na$, such a is central in G . Remembering that $n^* = n$ and $a^* \neq a$ implies $na = an$ (consider the commutativity of n and $a - a^*$), it follows that any $g \in G$ with $g^* = g$ is central. For instance, s is central in G .

Now fix $a \notin N$ with $a^* \neq a$. There exists $n \in N$ with $an \neq na$. Remember that this implies $na = an^*$. Also, n^{-1} and a cannot commute, so $(n^{-1})^* \neq n^{-1}$ and $n^{-1}a = a(n^{-1})^* = a(sn^{-1})$, so $s = a^{-1}n^{-1}an = (a, n)$, the commutator. We claim this implies that s is the only nonidentity commutator in G . This is already the case in the SLC group N so, noting that $G = N \cup Na$, there are just two types of commutators to consider. If $m, n \in N$, then $m(na) = (mn)a$ whereas

$$(na)m = nam = \begin{cases} nma & \text{if } am = ma \\ snma & \text{if } am \neq ma \end{cases}$$

and, since $nm = mn$ or $nm = smn$, it is clear that $(na)m = nma$ or $(na)m = snma$, so the commutator $(m, na) = 1$ or s . Similarly, it is easy to show that any commutator (na, m) with $n, m \in N$ is also 1 or s .

Again suppose that $na \neq an$ for some $n \in N$ and $a \notin N$. Then $(a, n) = s$. Also a^* and n cannot commute for $a^*n = na^*$ implies $a^*na = na^*a = a^*an$, which is false. So $s = (a^*, n)$ too. Now $na \neq (na)^*$; otherwise, $na = a^*n^* = a^*sn$, which says $san = a^*sn$ and $a = a^*$, which is not true. Thus $na - (na)^*$, which is $na - a^*n^* = na - sa^*n = na - na^* = n(a - a^*)$ and $a - a^*$ are each in \mathcal{T}_4 , so they commute. This gives

$$ana - ana^* - a^*na + a^*na^* = n(a^2 - 2aa^* + (a^*)^2),$$

which we rewrite as

$$sna^2 - snaa^* - sna^*a + sn(a^*)^2 = na^2 - 2naa^* + n(a^*)^2$$

and then

$$(3.5) \quad sna^2 + sn(a^*)^2 + 2naa^* = na^2 + n(a^*)^2 + 2snaa^*.$$

If $a^* \neq sa$, then $sna^2 \neq naa^*$ and naa^* appears on the left side of this equation while it cannot appear on the right. We conclude that $a^* = sa$ in which case (3.5) becomes $4sna^2 = 4na^2$, so R must have characteristic 4.

We have shown that if $a \notin N$ with $a^* \neq a$, then such an element is not central and $a^* = sa$. It follows that the involution on G is given by (1.1) and hence G is SLC: if $gh = hg$ and neither g nor h nor gh is central, then $sg h = (gh)^* = h^*g^* = (sh)(sg) = hg = gh$, a contradiction. The situation is as described in (4). □

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